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On a non-linear theory of thin jets. Part 1

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The injection of a two-dimensional jet into a uniform stream is considered, the fluids being assumed inviscid and incompressible. When the total head of the jet is much larger than that of the uniform flow, the motion is characterized by two disparate length scales, and uniformly valid asymptotic solutions can be found by the method of matched expansions. Inner and outer expansions are developed for the jet and the external flow. The first-order outer solution in the jet is the usual thin jet approximation which fails in the neighbourhood of the jet exit except for 90° injection, when it is uniformly valid. The basic non-linearity introduced by the pressure condition along the vortex sheet separating the jet from the external flow appears as a non-linear boundary condition for the first-order outer solution in the external flow. A novel feature of the analysis is the necessity of imposing a logarithmic singularity as an 'inner' boundary condition for the outer solution in the external flow. The first-order fluid speed and streamline deflection angle are shown to be given correctly to $O(1)$ uniformly in the external flow (for all injection angles) by the first-order outer solution.

1. Introduction

We will consider a jet of total head H_1 issuing from an infinite plate into a uniform flow of lower total head H_2 (see figure 1*a*). The fluid will be assumed

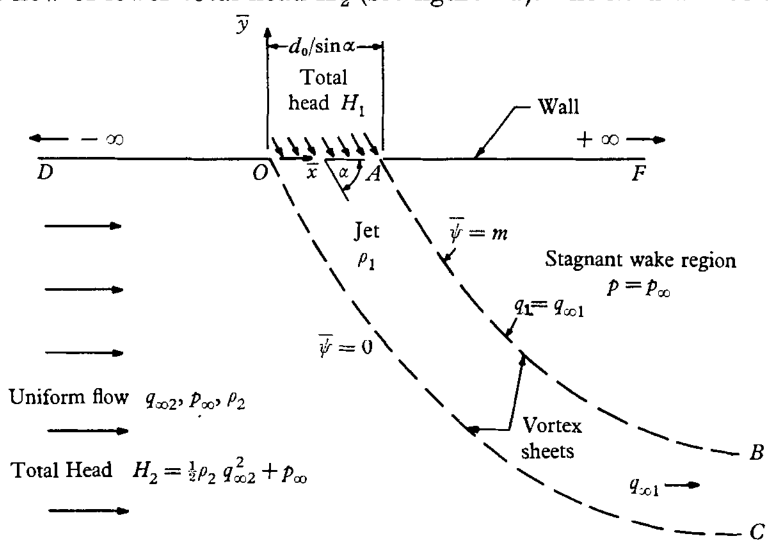

FIGURE 1*a*. Region of flow in physical plane.

inviscid and incompressible, and the region to leeward of the jet will be treated as a stagnant wake with constant pressure p_∞ equal to that of the undisturbed stream. Two-dimensional, steady, irrotational solutions of Euler's equations will be sought, but due to the difference in total heads, vortex sheets must separate the jet from the external flow if the static pressure is to be continuous throughout the fluid. Across the jet opening OA, the angle of injection $-\alpha$ is fixed, and at large distances from the origin all motion will be assumed to be directed in the positive $\bar{x}$-direction.

Perhaps the main difficulty in studying such flows is the non-linear boundary condition on the pressure which must be applied along a vortex sheet. Classical methods usually assume that the total head of the jet and the external flow are equal and in those cases the flow can be represented by a single complex velocity potential. However, problems of current engineering interest (e.g. VTOL aircraft, ground-effect machines, and jet-flapped wings) involve a large difference in total heads and to account properly for the jet free-stream interaction the basic non-linearity of the flow must be taken into account. In this paper the method of matched expansions is used to obtain uniformly valid asymptotic solutions in the jet and the external flow when $H_1/H_2 \to \infty$.

Taylor (1954) considered this problem for 90° injection and obtained a rough theoretical estimate for the shape of the jet using an approximate analysis. He pointed out that due to viscous spreading the jet would fill a wedge of nearly 40° and any experimental verification of theoretical results would have to be made close to the jet exit. In addition, the flow is very unstable and eddies would be formed in the (assumed) stagnant region behind the jet. However, since problems involving adjacent regions of different total heads occur frequently in nature and are inherently non-linear, their study is worthwhile and important in spite of these idealizations.

In some earlier work Ackerberg & Pal (1968) [hereafter referred to as A–P] studied this problem assuming that the ratio of the jet thickness d to its radius of curvature $\bar{R}$ would be small everywhere when $H_1 \gg H_2$. Using a thin jet approximation (see Preston 1954; Spence 1956) which relates the fluid speed inside the jet to its curvature, they derived a non-linear potential problem for the external flow which was solved numerically for 90° injection using a variational principle and a variant of the Ritz–Galerkin method. Ackerberg noted that the thin jet approximation would not satisfy the exact boundary condition at the jet exit except in the case of normal injection. This observation motivated this study which was to determine the regions of validity of the thin jet approximation and the solution of the aforementioned potential problem. The relationship between the physically imposable condition $H_1/H_2 \gg 1$ and the assumption $d/\bar{R} \ll 1$ was clarified, and the two shown to be equivalent if $\bar{R}$ is the radius of curvature of the jet centreline.

The identification of this problem as one involving a singular perturbation follows from an intuitive argument. Consider a point inside the jet at a fixed distance of $O(d_0)$ from the jet opening (see figure 1*b*, region III). When $H_1/H_2 \to \infty$, one expects the flow inclination at this point to approach the injection angle $-\alpha$. This limit will not be uniform at large distances (to be defined shortly) from the jet exit where both flows will be undeflected with respect to the uniform stream. Likewise, at a point in the adjacent external flow (region IV), the same limit must result in the flow in a corner of angle $(\pi - \alpha)$, which also will be non-uniform

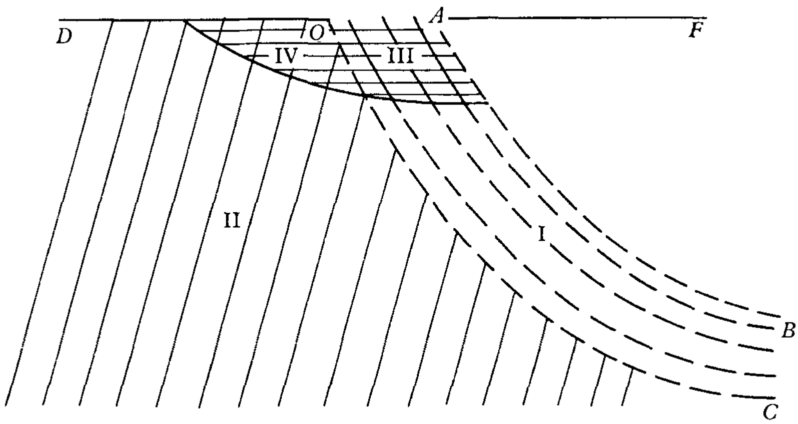

FIGURE 1*b*. Regions of different flow approximations.

at large distances. This non-uniform behaviour can be taken into account by finding solutions based on a length scale which is characteristic of the flow at large distances. These solutions must satisfy boundary conditions at infinity and will be required to merge with the solutions valid near the jet exit.

The length L which characterizes the flow at large distances can be deduced from a simple heuristic argument. Denote the jet's width and density, and the speed along the free streamline AB by d, ρ_1 and $q_{\infty 1}$, respectively, and let L be interpreted as the order of the length the jet must penetrate the external flow to obtain a change of $O(1)$ in the angular direction of the jet. Using an injection angle of 90° for simplicity, we note that as a result of the jet free-stream interaction over the distance L, the jet gains horizontal momentum of $O(\rho_1 d q_{\infty 1}^2)$ per unit time per unit breadth. This must equal the horizontal component of the pressure forces acting on the jet boundaries which will be of $O[(H_2 - p_\infty)L]$ per unit breadth.† Therefore, using Bernoulli's principle, we obtain:

$$(H_2 - p_\infty)L \equiv \tfrac{1}{2}\rho_2 q_{\infty 2}^2 L = O(\rho_1 d q_{\infty 1}^2), \tag{1.1}$$

and

$$L/d = O(\rho_1 q_{\infty 1}^2/\rho_2 q_{\infty 2}^2). \tag{1.2}$$

Thus, the outer length scale is much larger than the inner one when

$$\mu_1 = \rho_2 q_{\infty 2}^2/\rho_1 q_{\infty 1}^2 \to 0,$$

and solutions based on this scale (valid in regions I and II of figure 1*b*) will be referred to as outer solutions.

In § 3, a first-order outer solution is obtained in the jet which is equivalent to the thin jet approximation. It is shown that when $\alpha \neq \frac{1}{2}\pi$ this solution will not satisfy the boundary condition at the jet exit and an inner solution is necessary to eliminate this non-uniformity. In § 4, the outer solution in the jet is combined

† We have assumed that the pressure along OC is equal to the stagnation pressure of the external flow. Since this would tend to make the jet turn faster, L might be underestimated; but it is in fact given correctly by (1.2).

with Bernoulli's equation and a curvature condition to derive a non-linear boundary condition along the vortex sheet OC for the first-order outer solution in the adjacent external flow. This boundary condition is equivalent to the one derived by Taylor (1954) in a different way. The first-order outer solution in the external flow satisfies the non-linear potential problem first obtained by A–P. To avoid a trivial solution, which in this case corresponds to a uniform flow, a singularity must be imposed as an inner boundary condition. This ensures the turning of the external flow through an angle $-\alpha$ near the jet exit. Since this solution satisfies the boundary conditions which would be imposed on the exact solution (except possibly where the thin jet approximation fails), it is assumed (and verified *a posteriori*) to be uniformly valid to $O(1)$ in the external flow. With a method similar to one used by Clarke (1965) for the flow over a waterfall, an inner solution is constructed in the jet in § 5 using conformal mapping. Off-hand, this solution would be expected to modify the first-order outer solution in the external flow (which was assumed to be uniformly valid) through the boundary condition along OC. However, the modification only affects the curvature which is $O(\mu_1)$ and not the speed and deflexion which are of $O(1)$. A potential problem is formulated in § 6 for an inner solution in the external flow which corrects the streamline curvature. Finally in § 7, the results of this analysis are discussed and summarized.

2. Mathematical formulation

Introduce a co-ordinate system $\bar{z} = \bar{x} + i\bar{y}$, with origin at O (see figure 1*a*). Using the complex velocity potential $\bar{w} = \bar{\phi} + i\bar{\psi}$, the complex velocity in the usual notation is given by
$$d\bar{w}/d\bar{z} = \bar{u} - i\bar{v} = \bar{q}e^{-i\theta}.$$

Dimensional variables will be denoted by bars and, when necessary, variables for the jet and the external flow will be distinguished by the subscripts 1 and 2, respectively, to avoid confusion. Non-dimensionalize the co-ordinates, the complex velocity potentials, and the speeds in the following ways:
$$z_i = (q_{\infty i}/m)\bar{z}_i, \quad w_i = \bar{w}_i/m, \quad q_i = \bar{q}_i/q_{\infty i}, \quad (i = 1, 2), \tag{2.1}$$
where m is the volumetric flow rate per unit breadth in the jet and $q_{\infty i}\,(i = 1, 2)$ are the speeds at an infinite distance from the jet exit. In this notation the dimensionless complex velocities are
$$dw_i/dz_i = u_i - iv_i = q_i e^{-i\theta_i} \quad (i = 1, 2). \tag{2.2}$$
The flow region in the z-plane will be mapped on to the w-plane as shown in figure 2. The vortex sheet along OC requires the use of two velocity potentials, and the abscissa in the w-plane has been labelled accordingly. Thus, inside the jet
$$0 < \psi \leqslant 1 \quad \text{and} \quad \psi \cot\alpha \leqslant \phi_1 \leqslant \infty; \tag{2.3a}$$
and in the external flow
$$\psi < 0 \quad \text{and} \quad -\infty \leqslant \phi_2 \leqslant \infty. \tag{2.3b}$$
Since the locations of the streamlines OC and AB in the physical plane are not known *a priori*, it is convenient to formulate this problem in the w-plane using

the logarithm of the complex velocity,

$$\Gamma(w) = \ln(dw/dz) = Q(\phi,\psi) - i\theta(\phi,\psi). \tag{2.4}$$

By the usual arguments the real and imaginary parts of $\Gamma(w)$ satisfy the Cauchy–Riemann equations and are conjugate harmonic functions in each region. Thus,

$$\partial Q/\partial\phi = -\partial\theta/\partial\psi, \tag{2.5a}$$

$$\partial Q/\partial\psi = \partial\theta/\partial\phi, \tag{2.5b}$$

and

$$\nabla^2 Q = 0, \quad \nabla^2\theta = 0, \tag{2.6}$$

where $\nabla^2 = \partial^2/\partial\psi^2 + \partial^2/\partial\phi^2$. Henceforth a subscript on the dependent variables Q and θ will imply the appropriate independent variables (ϕ,ψ) which will not be subscripted.

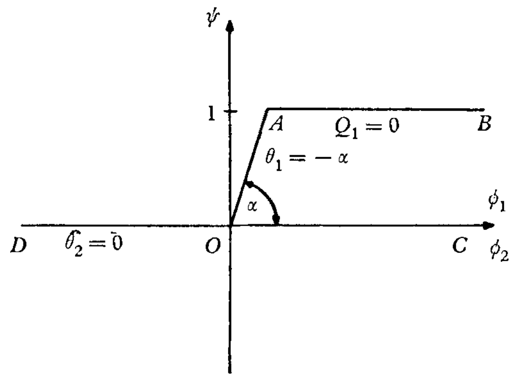

FIGURE 2. w_1- and w_2-planes.

Boundary conditions

Along OA and DO the deflexion is fixed. Thus,

$$\theta_1 = -\alpha \quad \text{for} \quad \phi = \psi\cot\alpha \quad (0 \leqslant \psi \leqslant 1), \tag{2.7a}$$

and

$$\theta_2 = 0 \quad \text{for} \quad -\infty \leqslant \phi < 0 \quad (\psi = 0). \tag{2.7b}$$

Since it will be most convenient to formulate the boundary value problems in terms of Q, (2.7a,b) will be expressed in terms of Q using the Cauchy–Riemann equations. Both (2.7a,b) may be written

$$\hat{\mathbf{t}}\cdot\nabla\theta = 0, \tag{2.8}$$

where the gradient is with respect to (ϕ,ψ) and $\hat{\mathbf{t}}$ is a unit vector tangent to DO or OA. Thus we may write (2.7a,b)

$$-\sin\alpha\,\partial Q_1/\partial\phi_1 + \cos\alpha\,\partial Q_1/\partial\psi = 0 \quad \text{for} \quad \phi = \psi\cot\alpha, \quad 0 \leqslant \psi \leqslant 1, \tag{2.9}$$

and

$$\partial Q_2/\partial\psi = 0 \quad \text{for} \quad -\infty \leqslant \phi < 0, \quad \psi = 0. \tag{2.10}$$

On crossing OC and AB the static pressure $\bar{p}$ must be continuous. The fixed wake pressure along AB requires that the speed just inside the jet be constant, and its value must be $q_{\infty 1}$. Therefore,

$$Q_1 = 0 \quad \text{for} \quad \cot\alpha \leqslant \phi \leqslant \infty, \quad (\psi = 1), \tag{2.11}$$

and along OC

$$\bar{p}_1 = \bar{p}_2 \quad \text{for} \quad \psi = 0, \tag{2.12}$$

where $\bar{p}_1$ and $\bar{p}_2$ must be evaluated at the same point in the $\bar{z}$-plane. Using Bernoulli's principle, (2.12) may be written

$$\rho_1 \bar{q}_1^2 - \rho_2 \bar{q}_2^2 = 2(H_1 - H_2) \quad \text{for} \quad \psi = 0. \tag{2.13}$$

At an infinite distance downstream where $q_i \to q_{\infty i}$, (2.13) becomes

$$(H_1 - H_2)/\tfrac{1}{2}\rho_1 q_{\infty 1}^2 = 1 - \rho_2 q_{\infty 2}^2/\rho_1 q_{\infty 1}^2 = 1 - \mu_1. \tag{2.14}$$

Substituting $\bar{q}_i/q_{\infty i} = e^{Q_i}$ and (2.14) into (2.13) yields

$$e^{2Q_1}|_{\phi=\phi_1(\sigma)} - \mu_1 e^{2Q_2}|_{\phi=\phi_2(\sigma)} = 1 - \mu_1 \quad \text{on} \quad \psi = 0, \tag{2.15}$$

where $\phi_1(\sigma) > 0$ and $\phi_2(\sigma) > 0$ refer to the same point on OC in the $\bar{z}$-plane with OC being given by $\bar{x} = \bar{x}(\sigma)$, $\bar{y} = \bar{y}(\sigma)$, σ being a parameter.

To guarantee that the fluid in the jet and the external flow remain contiguous, the deflexion on crossing the streamline OC must be continuous. Therefore, using the previous notation

$$\theta_1[\phi_1(\sigma), 0] = \theta_2[\phi_2(\sigma), 0]. \tag{2.16a}$$

Here it will also be useful to consider the continuity of the curvature $1/\bar{R} = \partial\theta/\partial\bar{s}$ on crossing OC. Noting that $\partial\theta/\partial\bar{s} = \bar{q}\partial\theta/\partial\bar{\phi}$, where $\bar{s}$ denotes arc-length along a streamline and $\bar{q} = \partial\bar{\phi}/\partial\bar{s}$, we obtain

$$q_{\infty 1}\left(e^{Q_1}\frac{\partial\theta_1}{\partial\phi}\right)\bigg|_{\phi=\phi_1(\sigma)} = q_{\infty 2}\left(e^{Q_2}\frac{\partial\theta_2}{\partial\phi}\right)\bigg|_{\phi=\phi_2(\sigma)} \quad \text{on} \quad OC.\dagger \tag{2.16b}$$

At large distances from the jet opening we require that the flow be undeflected and uniform in each region, i.e.

$$Q_1, \theta_1 \to 0 \quad \text{for} \quad |w_1| \to \infty, \tag{2.17a}$$

and

$$Q_2, \theta_2 \to 0 \quad \text{for} \quad |w_2| \to \infty. \tag{2.17b}$$

Finally, a stagnation point must be placed at point 0 in the external flow to ensure the turning of the fluid through an angle $-\alpha$. This requires

$$\Gamma_2 \sim \frac{\alpha}{\pi}\ln e^{\pi i} w_2 \quad \text{as} \quad w_2 \to 0, \tag{2.18}$$

where $0 \geqslant \arg w_2 \geqslant -\pi$. It is believed that (2.5)–(2.18) are sufficient to determine the two functions Γ_1 and Γ_2. However, the non-linear boundary conditions (2.15) and (2.16), which must be applied along OC whose position is unknown, make an exact treatment too difficult. Fortunately in most practical situations solutions are required when the total head of the jet is much larger than that of the external stream. In these cases asymptotic solutions can be found for $\mu_1 \to 0$.

3. The outer solution in the jet

When $d/\bar{R}_1 \ll 1$, it is expected that variations along the jet in the outer region (figure 1*b*, region I) will be much smaller than those across it, i.e. $\partial/\partial\phi_1 \ll \partial/\partial\psi$. This behaviour may be taken into account explicitly by altering the scale of ϕ_1. A closely related method is used in shallow water theory (Stoker 1957,

† In most practical cases (2.16*a*) and (2.16*b*) will be equivalent.

pp. 348–51), and it will be evident that the thin jet approximation represents the first term of an outer expansion in the terminology of singular perturbations.

In place of ϕ_1 introduce the new independent variable

$$\hat{\phi} = \mu_1 \phi_1, \tag{3.1}$$

which will remain of $O(1)$ in the outer region. To show that this scaling is correct, estimate $\bar{q}_1 = \partial\bar{\phi}_1/\partial\bar{s}$ in the outer region where $\bar{q}_1 = O(q_{\infty 1})$ and $\bar{s} = O(L)$; we find $\phi_1 = O(Lq_{\infty 1}/m) = O(1/\mu_1)$.† The thin jet approximation results formally by seeking solutions of the form

$$Q_1 \sim \mu_1 \hat{Q}(\hat{\phi}, \psi) + o(\mu_1), \tag{3.2a}$$

and

$$\theta_1 \sim \hat{\theta}(\hat{\phi}, \psi) + o(1), \tag{3.2b}$$

where $\sim$ is used to denote the asymptotic nature of these solutions valid for small μ_1; it is implied that $\hat{Q}$ and $\hat{\theta}$ remain of $O(1)$ in the region where $\hat{\phi}$ and ψ are of $O(1)$. Using $(3.2a,b)$ we find that the non-dimensional curvature $d/\bar{R}_1$ is of $O(\mu_1)$ and this agrees qualitatively with the assumption of a thin jet.‡

Substituting $(3.2a,b)$ into $(2.5a,b)$ which have been transformed to the variables $(\hat{\phi}, \psi)$, we find on equating the coefficients of each power of μ_1 to zero

$$\partial\hat{\theta}/\partial\psi = 0, \tag{3.3a}$$

and

$$\partial\hat{Q}/\partial\psi = \partial\hat{\theta}/\partial\hat{\phi}. \tag{3.3b}$$

Equation $(3.3a)$ yields the expected result that $\hat{\theta}$ does not vary across the jet thickness to first order. Thus,

$$\hat{\theta}(\hat{\phi}, \psi) \equiv \hat{\theta}(\hat{\phi}). \tag{3.4}$$

Using this result in $(3.3b)$ gives

$$Q(\phi, \psi) = \psi\theta'(\phi) + a(\phi), \tag{3.5}$$

where $a(\hat{\phi})$ is an arbitrary function and the prime has been used to denote differentiation. Applying the boundary condition (2.11) yields

$$a(\hat{\phi}) = -\hat{\theta}'(\hat{\phi}), \tag{3.6}$$

and

$$\hat{Q}(\hat{\phi}, \psi) = \hat{\theta}'(\hat{\phi})(\psi - 1). \tag{3.7}$$

The unknown function $\hat{\theta}(\hat{\phi})$ will be determined by applying the boundary conditions (2.15) and $(2.16b)$ which ensure the continuity of the static pressure and curvature on crossing $\psi = 0$. Equation (3.7) is related to the more usual form of the thin jet approximation in the appendix §1.

It is of interest to try to impose the boundary condition (2.9) along OA assuming that $(3.2a,b)$ may be used at the jet exit. Transforming (2.9) to the variables

† In the jet $m = O(dq_{\infty 1})$.

‡ Equation $(3.2a)$ also implies that the jet thickness remains constant to first order, i.e.

$$d = \int_0^m d\bar{\psi}/\bar{q}_1 = (m/q_{\infty 1})\int_0^1 e^{-Q_1} d\psi = (m/q_{\infty 1})\int_0^1 \exp\{-\mu_1\hat{Q} + \ldots\}d\psi = m/q_{\infty 1} + o(1).$$

The higher-order terms in these expansions (and others which will be given later) are likely to involve fractional powers and logarithms of μ_1. Here we will be primarily concerned with the first-order solutions.

$(\hat{\phi},\psi)$ and substituting (3.2a), we find on equating the coefficient of μ_1 to zero,

$$\cos\alpha\,\partial\hat{Q}/\partial\psi = 0 \quad \text{for} \quad \hat{\phi} = \mu_1\psi\cot\alpha, \quad (0 \leqslant \psi \leqslant 1). \tag{3.8}$$

For $\alpha \neq \frac{1}{2}\pi$ this requires

$$\hat{\theta}'(\hat{\phi}) = 0 \quad \text{for} \quad \hat{\phi} = \mu_1\psi\cot\alpha, \quad (0 \leqslant \psi \leqslant 1), \tag{3.9}$$

which can only be satisfied if $\hat{\theta}(\hat{\phi}) \equiv -\alpha$ in accordance with (2.7a). This, however, violates the boundary condition at infinity (2.17a). Such a paradox is not surprising since the expansions (3.2a,b) are valid for $\hat{\phi} = O(1)$, and in the $(\hat{\phi},\psi)$-plane (3.8) is applied where $\hat{\phi} = O(\mu_1)$, i.e. when $\phi_1 = O(1)$. To satisfy this boundary condition an inner expansion based on the independent variables (ϕ_1,ψ) must be used.

When $\alpha = \pi/2$ (3.8) is satisfied, and on equating the coefficient of μ_1^2 to zero we obtain

$$\partial\hat{Q}/\partial\hat{\phi} = 0 \quad \text{for} \quad \hat{\phi} = 0, \quad (0 \leqslant \psi \leqslant 1), \tag{3.10}$$

which requires

$$\hat{\theta}''(0) = 0. \tag{3.11}$$

It is shown in the appendix § 2, that (3.11) is indeed satisfied. Thus, it is possible to use the thin jet approximation up to the jet opening only when $\alpha = \frac{1}{2}\pi$.

4. The outer solution in the external flow

The correct scaling for the velocity potential in the outer region of the external flow can be deduced as in § 3. Noting that in this region $\bar{q}_2 = O(q_{\infty 2})$ and $\bar{s} = O(L)$, we readily deduce $\phi_2 = O(q_{\infty 2}/q_{\infty 1}\mu_1)$. Therefore, introduce the non-dimensional parameter

$$\mu_2 = \rho_2 q_{\infty 2}/\rho_1 q_{\infty 1} = (q_{\infty 1}/q_{\infty 2})\mu_1,\dagger \tag{4.1}$$

and the scaled velocity potential

$$\tilde{\phi} = \mu_2\phi_2. \tag{4.2}$$

In this region where we expect Q_2 and θ_2 to remain of $O(1)$ it is necessary to scale ψ also; otherwise, there would be no dependence on ψ to first order and this would not satisfy the boundary conditions. Thus, define

$$\tilde{\psi} = \mu_2\psi \quad \text{for} \quad \psi < 0, \tag{4.3}$$

and assume expansions of the form

$$Q_2 \sim \tilde{Q}(\tilde{\phi},\tilde{\psi}) + o(1), \tag{4.4a}$$

and

$$\theta_2 \sim \tilde{\theta}(\tilde{\phi},\tilde{\psi}) + o(1). \tag{4.4b}$$

$\tilde{Q}$ and $\tilde{\theta}$ satisfy the Cauchy–Riemann equations (2.5a,b) with (ϕ,ψ) replaced by $(\tilde{\phi},\tilde{\psi})$; thus,

$$\nabla^2\tilde{Q} = 0, \quad \nabla^2\tilde{\theta} = 0, \tag{4.5}$$

where now $\nabla^2 = \partial^2/\partial\tilde{\phi}^2 + \partial^2/\partial\tilde{\psi}^2$.

Boundary conditions

Transforming (2.10) we find

$$\partial\tilde{Q}/\partial\tilde{\psi} = 0 \quad \text{for} \quad \tilde{\phi} < 0, \quad \tilde{\psi} = 0. \tag{4.6}$$

† Throughout this paper it will be assumed that $(\rho_2/\rho_1) = O(1)$, so that $\mu_1 \propto \mu_2^2$. We will use μ_1 and μ_2 as necessary to avoid writing (ρ_2/ρ_1).

Substituting (3.2a) and (4.4a) into (2.15), expanding for small μ_1, and equating the coefficient of μ_1 to zero yields

$$2\hat{Q}(\hat{\phi},0) - \exp\{2\tilde{Q}(\tilde{\phi},0)\} = -1 \quad \text{for} \quad \hat{\phi},\tilde{\phi} > 0. \tag{4.7}$$

Using (3.7) this may be written

$$2\hat{\theta}'(\hat{\phi}) + \exp\{2\tilde{Q}(\tilde{\phi},0)\} = 1 \quad \text{for} \quad \hat{\phi},\tilde{\phi} > 0. \tag{4.8}$$

Transforming (2.16b) to the variables $(\hat{\phi},\tilde{\phi})$, substituting the expansions (3.2a,b) and (4.4a,b) and expanding for small μ_1, we find on using (4.1) and equating the largest coefficient of μ_1, to zero

$$\hat{\theta}'(\hat{\phi}) = \exp\{\tilde{Q}(\tilde{\phi},0)\}(\partial\tilde{\theta}/\partial\tilde{\phi}) \quad \text{for} \quad \tilde{\psi} = 0,\, \hat{\phi},\tilde{\phi} > 0. \tag{4.9}$$

Eliminating $\hat{\theta}'$ between (4.8) and (4.9) yields

$$\partial\tilde{\theta}/\partial\tilde{\phi} = \partial\tilde{Q}/\partial\tilde{\psi} = -\sinh\tilde{Q}(\tilde{\phi},0) \quad \text{for} \quad \tilde{\psi} = 0,\, \tilde{\phi} > 0. \tag{4.10}$$

At large distances (2.17b) requires

$$\tilde{Q} \to 0 \quad \text{as} \quad |\tilde{w}| \to \infty, \tag{4.11}$$

where $\tilde{w} = \mu_2 w_2$.

To complete the formulation of problems similar to that for $\tilde{Q}$, a matching condition is usually required when $|\tilde{w}| \to 0$. If w^+ denotes the stretched variable for the inner solution in the external flow, the matching condition is obtained by letting $|w^+| \to \infty$ in the inner solution and afterward expressing what remains (neglecting exponentially small terms) in terms of $\tilde{w}$. In most cases the remainder will involve positive powers of $\tilde{w}$ so that $\tilde{Q}$ remains bounded at $\tilde{w} = 0$.† A harmonic function which is bounded at $\tilde{w} = 0$ and satisfies (4.6), (4.10) and (4.11) is $\tilde{Q} \equiv 0$. This predicts $\bar{q}_2 \equiv q_{\infty 2}$ everywhere, and usually such a trivial solution indicates improper scaling. If, however, the whole external flow is characterized by the single length L to lowest order, it would be correct to apply the singular boundary condition (2.18) with w_2 replaced by $\tilde{w}$, i.e.

$$\tilde{Q} \sim (\alpha/\pi)\ln|\tilde{w}| \quad \text{for} \quad \tilde{w} \to 0. \tag{4.12}$$

This would ensure the turning of the external flow through the angle $-\alpha$ near the jet exit and the trivial solution would be avoided. Pal (1965) has proved that a solution satisfying (4.12) and the other boundary conditions exists and is unique. Since a solution may be found which satisfies all the boundary conditions of the exact formulation in § 2 (except possibly for a small region adjacent to OC where the thin jet approximation fails), we expect this solution to be uniformly valid to first order throughout the external flow. We now postulate this to be the case, and will show *a posteriori* that inner solutions in the jet and the external flow can be found in a consistent way.

Some interesting deductions about $\tilde{Q}$ can be made from the boundary conditions without the benefit of the numerical solution obtained by A–P. Letting $\tilde{q} = e^{\tilde{Q}}$, (4.10) may be written

$$1/\tilde{R} \equiv \tilde{q}(\partial\tilde{\theta}/\partial\tilde{\phi}) = \tfrac{1}{2}(1-\tilde{q}^2) \quad \text{for} \quad \tilde{\psi} = 0,\, \tilde{\phi} > 0. \tag{4.13}$$

This shows that along OC the non-dimensional curvature and non-dimensional pressure are equal to first order as Taylor (1954) observed. If $\tilde{q}$ exceeded unity

† This assumes, of course, that the domains of validity of these solutions overlap. At this point there is no reason to believe otherwise.

along OC, the curvature would be negative and the jet would bend upstream, which is an unlikely physical result; in fact, Pal (1965) has proved that $\tilde{q}$ increases monotonically along OC from a zero-value at O to the unit value an infinite distance downstream.

Asymptotic expansions of $\tilde{\Gamma}$

Formal asymptotic expansions for $\tilde{\Gamma}(\tilde{w}) = \tilde{Q} - i\tilde{\theta}$ were found by A–P for $|\tilde{w}| \to 0$ and $|\tilde{w}| \to \infty$. For $|\tilde{w}| \to 0$ the first few terms in this expansion are

$$\tilde{\Gamma}(\tilde{w}) \sim \frac{\alpha}{\pi}\ln(e^{\pi i}\tilde{w}) + a_0 + a_1(e^{\pi i}\tilde{w})^{1-(\alpha/\pi)} + a_2(e^{\pi i}\tilde{w}) + \ldots \quad \text{for} \quad |\tilde{w}| \to 0, \tag{4.14}$$

where the a_i's are real and $0 \geqslant \arg \tilde{w} \geqslant -\pi$. a_0 and a_2 cannot be determined by this formal procedure and

$$a_1 = -\tfrac{1}{2}e^{-a_0}[(1-\alpha/\pi)\sin\alpha]^{-1}. \tag{4.15}$$

When $|\tilde{w}| \to \infty$,

$$\tilde{\Gamma}(\tilde{w}) \sim \frac{c_0}{(e^{\pi i}\tilde{w})^{\frac{1}{2}}} + \frac{c_1\ln(e^{\pi i}\tilde{w}) + c_2}{(e^{\pi i}\tilde{w})^{\frac{3}{2}}} + \ldots, \tag{4.16}$$

where c_0 and c_2 are indeterminate by this formal procedure and

$$c_1 = -c_0/2\pi. \tag{4.17}$$

5. The inner solution in the jet

The necessity of finding an inner solution in the jet was shown in §3 where it was found that the thin jet approximation could not satisfy the boundary condition at the jet exit unless $\alpha = \frac{1}{2}\pi$. In the inner region d_0 will be the characteristic length and to first order the jet is expected to remain undeflected. Since the increase in static pressure near O due to the stagnation of the external stream must be small compared to the total head of the jet when $\mu_1 \ll 1$, the velocity variation across the jet will be small; thus $\bar{q}_1 \sim q_{\infty 1}$, and $Q_1 = o(1)$. Using these estimates, we easily find that (ϕ_1, ψ) are the natural co-ordinates in this region. Translating these ideas into mathematical form leads to the following expansions:

$$Q_1 \sim \mu_1 Q^*(\phi_1, \psi) + o(\mu_1), \tag{5.1a}$$

and

$$\theta_1 \sim -\alpha + \mu_1\theta^*(\phi_1, \psi) + o(\mu_1), \tag{5.1b}$$

where Q^* and θ^* are conjugate harmonic functions which are related by the Cauchy–Riemann equations $(2.5a,b)$. From $(5.1a,b)$ it is evident that the jet thickness remains constant $(= d_0)$ to first order† and the non-dimensional curvature $d/\bar{R}_1$ is again of $O(\mu_1)$; however, this last conclusion will require modification later on.

Boundary conditions

The condition $(2.7a)$ at the jet exit requires

$$\theta^*(\phi_1, \psi) = 0 \quad \text{for} \quad \phi_1 = \psi\cot\alpha, \quad (0 \leqslant \psi \leqslant 1). \tag{5.2}$$

Using (2.8) and the Cauchy–Riemann equations, this may be written

$$\partial Q^*/\partial n = 0 \quad \text{for} \quad \phi_1 = \psi\cot\alpha, \quad (0 \leqslant \psi \leqslant 1), \tag{5.3}$$

† Assuming that the domain of validity of these solutions overlaps with that of the thin jet approximation, it is apparent that the jet thickness will remain constant to first order throughout its course.

where $\partial/\partial n$ denotes the derivative normal to the line OA. Along the free streamline AB (2.11) requires

$$Q^*(\phi_1, 1) = 0 \quad \text{for} \quad \cot\alpha \leqslant \phi_1 \leqslant \infty. \tag{5.4}$$

The static pressure must be continuous on crossing OC. Since $\tilde{\Gamma}$ has been assumed to be a uniformly valid first approximation in the external flow, we may use it to compute the pressure along OC. However, when $\tilde{w}$ was introduced as a natural variable in the external flow it was assumed to remain of $O(1)$ when $\bar{z}_2 = O(L)$. Near the jet exit where $\bar{z}_2 = O(d_0)$, we will have $\tilde{w} = o(1)$; thus, it is sufficient to use the asymptotic expansion (4.14) to compute the pressure correctly to first order via the velocity and Bernoulli's equation. To obtain an expression for $\tilde{q}$, substitute (4.14) into (2.4) and integrate, choosing $\tilde{w} = 0$ when $z_2 = 0$. We find

$$\tilde{w} \propto (\mu_2 z_2)^{\pi/(\pi-\alpha)}. \tag{5.5}$$

Substituting (5.5) into (4.14) yields for $\tilde{w} \to 0$

$$\tilde{q} = e^{\tilde{Q}} \propto |\tilde{w}|^{\alpha/\pi} \propto (\mu_2 z_2)^{\alpha/(\pi-\alpha)}. \tag{5.6}$$

Near the jet exit $\bar{z}_2 = O(d_0)$, and $\mu_2 z_2 \equiv (\mu_2 q_{\infty 2}/m)\bar{z}_2 = O(\mu_1)$. Thus,

$$\tilde{q} = e^{\tilde{Q}} = O[\mu_1{}^{\alpha/(\pi-\alpha)}] = o(1), \tag{5.7}$$

and from Bernoulli's equation the pressure along OC in this region is constant to first order and equal to the stagnation pressure of the external stream. Substituting (5.1a) and (4.4a), using (5.7), into (2.15), and expanding for $\mu_1 \to 0$ yields

$$1 + 2\mu_1 Q^* + o(\mu_1) = 1 - \mu_1 \quad \text{on} \quad \psi = 0, \phi_1 > 0. \tag{5.8}$$

Therefore,

$$Q^*(\phi_1, 0) = -\tfrac{1}{2} \quad \text{for} \quad \phi_1 > 0. \tag{5.9}$$

This boundary condition, which is crucial for obtaining the inner solutions in the jet and the external flow, depends only on the order relation (5.7) and not on the detailed analytic behaviour of $\tilde{\Gamma}$.

The boundary conditions (5.3), (5.4), (5.9) and a boundedness condition at infinity are sufficient to determine a harmonic function Q^* in the strip $COAB$. θ^*, the conjugate harmonic function, may be determined from Q^* except for an arbitrary constant.

To solve for Q^*, map the strip $COAB$ in figure 3, on to the upper half of the ζ-plane in figure 4 using a Schwarz–Christoffel transformation. This result is

$$w_1 = \frac{1}{\pi}\int_1^\zeta \left(\frac{t-1}{t+1}\right)^{\alpha/\pi} \frac{dt}{t-1}, \tag{5.10}$$

where $\operatorname{Im}\zeta \geqslant 0$ and the path of integration is chosen so that $\operatorname{Im} t \geqslant 0$ with $0 \leqslant \arg(t-1), \arg(t+1) \leqslant \pi$. When $\alpha = \frac{1}{2}\pi$, (5.10) can be evaluated exactly to yield

$$w_1 = \frac{1}{\pi}\ln[\zeta + (\zeta^2 - 1)^{\frac{1}{2}}] \quad \text{for} \quad \alpha = \tfrac{1}{2}\pi, \tag{5.11}$$

where $0 \leqslant \arg(\zeta - 1)$, $\arg(\zeta+1) \geqslant \pi$. The mapping (5.11) can also be used to map the s-plane (figure 5) on to the ζ-plane, i.e.

$$s = \sigma + i\delta = \frac{1}{\pi}\ln[\zeta + (\zeta^2-1)^{\frac{1}{2}}]. \tag{5.12}$$

The solution for Q^* can be found by inspection in the s-plane; it is

$$Q^*(\sigma,\delta) = -\tfrac{1}{2}(1-\delta) = -\tfrac{1}{2}\operatorname{Re}(1+is). \tag{5.13}$$

Thus, the solution for $\Gamma^* = Q^* - i\theta^*$ is given by

$$\Gamma^*(\zeta) = -\tfrac{1}{2}\{1+\frac{i}{\pi}\ln[\zeta+(\zeta^2-1)^{\frac{1}{2}}]\}, \tag{5.14}$$

and an imaginary constant has been chosen to satisfy (5.2).

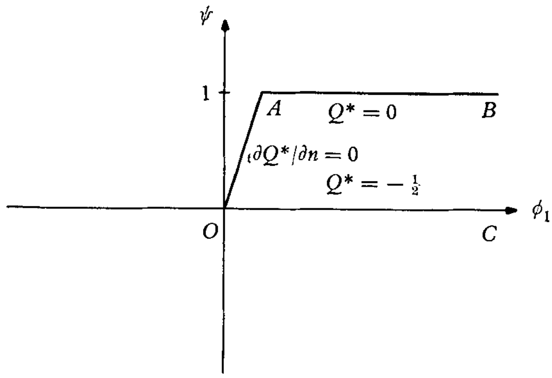

FIGURE 3. w_1-plane.

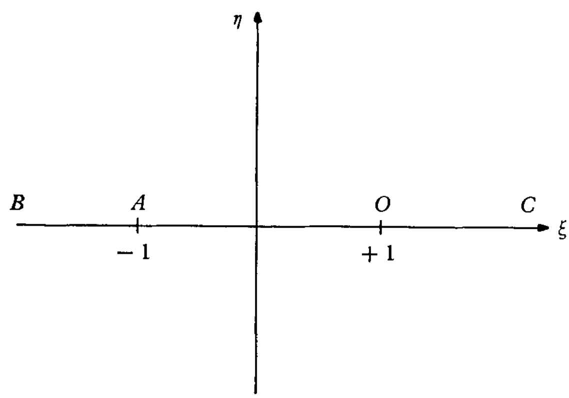

FIGURE 4. ζ-plane.

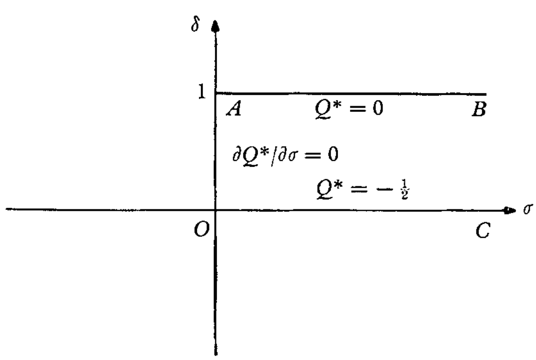

FIGURE 5. s-plane.

The behaviour of $\Gamma^*(w_1)$ for $|w_1| \to \infty$ is readily found. From (5.10)

$$w_1(\zeta) \sim \frac{1}{\pi}\ln\zeta + C - \frac{1}{\pi}\left(1 - \frac{2\alpha}{\pi}\right)(1/\zeta) + O(1/\zeta^2) \quad \text{for} \quad |\zeta| \to \infty, \tag{5.15}$$

where

$$C = \frac{1}{\pi}\int_0^1 \left\{\left(\frac{1-u}{1+u}\right)^{\alpha/\pi} \frac{1}{1-u} - 1\right\}\frac{du}{u}. \tag{5.16}$$

Inverting (5.15) for $\zeta(w_1)$ we find

$$\zeta \sim e^{\pi(w_1 - C)} + O(e^{-\pi(\phi_1 - C)}) \quad \text{as} \quad |w_1| \to \infty. \tag{5.17}$$

Using (5.17) in (5.14) which has been expanded for large ζ yields

$$\Gamma^*(w_1) \sim -(\tfrac{1}{2})[1 + (i/\pi)\ln 2 + i(w_1 - C) + O(e^{-2\pi\phi_1})]. \tag{5.18}$$

Therefore,

$$Q^* \sim -(\tfrac{1}{2})(1-\psi) \quad \text{as} \quad |w_1| \to \infty, \tag{5.19}$$

and

$$\theta^* \sim (\tfrac{1}{2})[(1/\pi)\ln 2 - C + \phi_1] \quad \text{as} \quad |w_1| \to \infty; \tag{5.20}$$

note that the error in these equations is exponentially small. When (5.20) is substituted in (5.1*b*) and the result expressed in terms of the outer variable $\hat{\phi}$ we find

$$\theta_1 \sim -\alpha + (\tfrac{1}{2})\hat{\phi} + (\tfrac{1}{2}\mu_1)[(1/\pi)\ln 2 - C]. \tag{5.21}$$

This must be the asymptotic behaviour of (3.2*b*) when $\hat{\phi} \to 0$. Therefore, unless $C = (1/\pi)\ln 2$ (which is the case for $\alpha = \frac{1}{2}\pi$) it is likely that the next term in (3.2*b*) is of $O(\mu_1)$.

Streamline curvature

The curvature of any streamline to first order is given by

$$d/\bar{R}_1(\phi_1, \psi) = \mu_1 \operatorname{Re}(id\Gamma^*/dw_1) = \mu_1 \operatorname{Re}[i(d\Gamma^*/d\zeta)(d\zeta/dw_1)] \tag{5.22}$$

$$= (\mu_1/2)\operatorname{Re}\left(\frac{\zeta-1}{\zeta+1}\right)^{\frac{1}{2}-(\alpha/\pi)}. \tag{5.23}$$

Near the point O where $\zeta \approx 1$, (5.10) may be solved for $\zeta(w_1)$, i.e.

$$\zeta - 1 = 2(\alpha w_1)^{\pi/\alpha} + \frac{2\alpha}{\pi+\alpha}(\alpha w_1)^{2\pi/\alpha} + \ldots, \tag{5.24}$$

where $0 \leqslant \arg w_1 \leqslant \alpha$. Substituting this result in (5.23) we obtain

$$d/\bar{R}_1(\phi_1, \psi) \sim (\mu_1/2)\operatorname{Re}[(\alpha w_1)^{(\pi/2\alpha)-1} + O(w_1)^{(3\pi/2\alpha)-1}], \tag{5.25}$$

and along OC where $w_1 = \phi_1$

$$d/\bar{R}_1(\phi_1, 0+) = (\mu_1/2)(\alpha\phi_1)^{(\pi/2\alpha)-1} + \ldots \quad \text{for} \quad \phi_1 \ll 1. \tag{5.26}$$

Thus, the curvature of OC is zero, finite, or infinite at O depending on whether $\alpha < \frac{1}{2}\pi$, $\alpha = \frac{1}{2}\pi$, or $\alpha > \frac{1}{2}\pi$. For $\alpha > \frac{1}{2}\pi$, the thin jet approximation $d/\bar{R}_1 \ll 1$ is not equivalent to the physically imposable condition $\mu_1 \to 0$ if $\bar{R}_1$ is taken along OC where $\phi_1 = O(\mu_1^{2\alpha/(2\alpha-\pi)})$. For the equivalence to remain valid $d/\bar{R}_1$ should always be based on the jet centre line $\psi = O(1)$. For $\alpha = \frac{1}{2}\pi$, (5.11) and (5.14) yield $\Gamma^*(w_1) \equiv -\frac{1}{2}(1 + iw_1)$. Using (5.22), $d/\bar{R}_1 \equiv \mu_1/2 + o(\mu_1)$ throughout the inner region of the jet.

The asymptotic solution for Γ^* [see (5.18)] can be used to compute the first-order jet curvature for large ϕ_1. We find

$$d/\bar{R}_1 = \tfrac{1}{2}\mu_1 + o(\mu_1) \quad \text{for} \quad 0 \leqslant \psi \leqslant 1. \tag{5.27}$$

Matching will require this result to agree with the first-order curvature given by the thin jet approximation when $\hat{\phi} \to 0$. Using (3.2) and (3.4) the curvature in the thin jet region is

$$d/\bar{R}_1 = \mu_1 \hat{\theta}'(\hat{\phi}) + o(\mu_1),$$

which does not vary across the jet to first order; thus the first-order curvature when $\hat{\phi} \to 0$ can be computed using $\tilde{\Gamma}$ along OC for $\tilde{\phi} \to 0$.† Expressing (4.13) in dimensional form and using (5.7) yields

$$d/\bar{R}_2 = \tfrac{1}{2}\mu_1 + o(\mu_1) \quad \text{for} \quad \tilde{\psi} = 0,\ \tilde{\phi} \to 0+. \tag{5.28}$$

The agreement between (5.27) and (5.28) is a strong indication that our method is correct.

The important boundary condition (5.9) was derived from the requirement that the static pressure be continuous on crossing OC. The condition that the jet and the external flow remain contiguous along OC in this region is also satisfied to first order as can be seen from (4.14) with $\tilde{\psi} = 0$, $\tilde{\phi} > 0$, i.e.

$$\tilde{\theta} = -\alpha + O(|\tilde{w}|^{1-(\alpha/\pi)}) = -\alpha + O(\mu_1). \tag{5.29}$$

Although the second term in (5.29) is of the same order as the second term in (5.1b), it would be extremely fortuitous if these terms were equal all along OC because θ^* does not depend on the details of $\tilde{\Gamma}$ as mentioned before. For this reason an inner solution will be required in the external flow to ensure the contiguity of the jet and the free stream to second order.

Finally, a uniformly valid approximation to Q_1 can be obtained by combining $\hat{Q}$ and Q^* to form the composite series [see Van Dyke (1965)]:

$$\begin{aligned} Q_1(\phi_1, \psi; \mu_1) &\sim \mu_1[\hat{Q}(\hat{\phi}, \psi) + Q^*(\phi_1, \psi) - Q^*(\infty, \psi)] + o(\mu_1) \\ &\sim \mu_1[\hat{Q}(\hat{\phi}, \psi) + Q^*(\phi_1, \psi) + \tfrac{1}{2}(1-\psi)] + o(\mu_1) \\ &\qquad \text{for } 0 \leqslant \phi_1, \hat{\phi} \leqslant \infty, 0 \leqslant \psi \leqslant 1. \end{aligned} \tag{5.30}$$

6. The inner solution in the external flow

Although $\tilde{\Gamma}$ will turn out to be a uniformly valid first approximation correct to $O(1)$ in the external flow, there is no reason to expect that the streamline curvature computed from it (which depends on $d\tilde{\Gamma}/d\tilde{w}$) will also be uniformly valid to first order.‡ In the inner region $\tilde{\Gamma}$ will not, in general, give correct values for the streamline curvature, and to ensure the contiguity of the jet and the external flow to second order [cf. (5.1b) and (5.29)] an inner solution will be required. The inner solution must contain all terms of $O(1)$ or larger which

† Since $\hat{\theta}(\hat{\phi})$ is unknown the jet curvature must be found by this roundabout method. The curvature computed in this way from $\tilde{\Gamma}$ must agree with that of the thin jet approximation along OC because the boundary condition (2.16b) guarantees this.

‡ A classical example which illustrates this behaviour is the Stokes solution for the flow past a sphere. The velocity components are uniformly valid to $O(1)$, but the velocity derivatives at large distances are not. This prevented Whitehead from improving Stokes' result.

appear in $\tilde{\Gamma}$ when $|\tilde{z}_2| = O(d_0)$. By inspection, these terms are the first two in the asymptotic expansion (4.14). Therefore, assume

$$\Gamma_2 \sim (\alpha/\pi)\ln(e^{\pi i}\tilde{w}) + a_0 + \mu_1\Gamma^+(w^+) + o(\mu_1), \tag{6.1}$$

where $\Gamma^+(w^+) = Q^+ - i\theta^+$, and Q^+ and θ^+ are conjugate harmonic functions of (ϕ^+, ψ^+). The mixed notation $(\tilde{w}, w^+)$ is used in (6.1) because the relationship

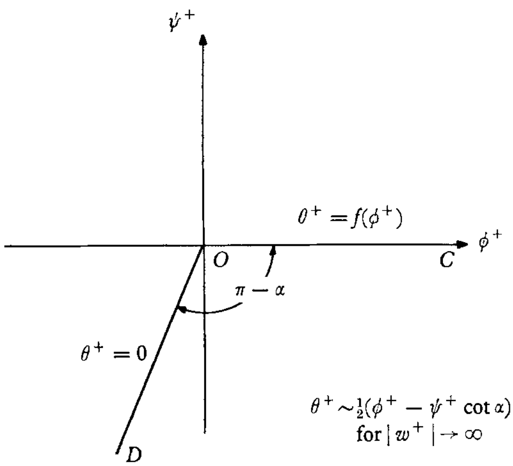

FIGURE 6. w^+-plane.

between w^+ and w_2 is unknown. If $|\Gamma^+|$ and $|w^+|$ are of $O(1)$ when $|w_1| = O(1)$, it should be possible to find a θ^+ which is equal to θ^* along OC. We note that the first two terms of (6.1) satisfy the boundary conditions (2.7*b*) and (2.18), and on applying (2.12), the boundary condition (5.9) is unchanged; thus, the inner solution in the jet is unaltered to $O(\mu_1)$ by the inner solution in the external flow.

To deduce the correct scaling in this region, note that $\tilde{q} = O(\mu_1^{\alpha/(\pi-\alpha)})$ from (5.7), and $\bar{s} = O(d_0)$; thus,

$$\phi_2 = O(q_{\infty 2} d_0 \mu_1^{\alpha/(\pi-\alpha)}/m) = O(\mu_2^{(\pi+\alpha)/(\pi-\alpha)}).$$

Therefore, introduce the new variable

$$w^+ = \phi^+ + i\psi^+ = \frac{e^{-a_0}\mu_2^{1-(\alpha/\pi)}}{[1-(\alpha/\pi)]\mu_1} w_2^{1-(\alpha/\pi)},\dagger \tag{6.2}$$

where $0 \geqslant \arg w^+ \geqslant -(\pi-\alpha)$. Along OC where $w_2 = \phi_2 = O(\mu_2^{[\pi+\alpha]/[\pi-\alpha]})$ [see (5.5)] we will have $w^+ = \phi^+ = O(1)$. This transformation maps the lower half of the w_2-plane onto the wedge-like region in the w^+-plane shown in figure 6. Since (6.2) is a conformal transformation at every finite point except the origin,

$$\nabla^2 Q^+ = 0, \quad \nabla^2\theta^+ = 0, \tag{6.3}$$

where now $\nabla^2 = \partial^2/\partial\psi^{+2} + \partial^2/\partial\phi^{+2}$. Hereafter it is easiest to formulate a boundary-value problem for θ^+.

† The multiplicative factors of $O(1)$ are introduced for convenience later on.

Boundary conditions

Along DO the deflexion is zero. Therefore,

$$\theta^+(\phi^+,\psi^+) = 0 \quad \text{for} \quad \phi^+ = \psi^+ \cot\alpha, \ (\psi^+ < 0). \tag{6.4}$$

On crossing OC the deflexion must be continuous. Hence, from (2.16a) and (5.1b)

$$\theta^+[\phi^+(\sigma),0] = \theta^*[\phi_1(\sigma),0] \quad \text{for} \quad \phi^+(\sigma),\ \phi_1(\sigma) > 0, \tag{6.5}$$

where OC is given parametrically by $\bar{x} = \bar{x}(\sigma), \bar{y} = \bar{y}(\sigma)$. Along OC, $w_1 = \phi_1$ and ζ is real and $\geqslant 1$. Thus, (5.14) becomes

$$\theta^*(\phi_1,0) = \frac{1}{2\pi}\ln[\zeta + (\zeta^2-1)^{\frac{1}{2}}], \tag{6.6}$$

where ζ is related to ϕ_1 through (5.10), i.e.

$$\phi_1(\zeta) = \frac{1}{\pi}\int_1^{\zeta}\left(\frac{t-1}{t+1}\right)^{\alpha/\pi}\frac{dt}{t-1}. \tag{6.7}$$

Equations (6.6) and (6.7) may be written in the simpler form

$$\theta^*(\phi_1,0) = f(\phi_1) \quad \text{for} \quad \phi_1 > 0, \tag{6.8}$$

where $f(\phi_1)$ may be assumed to be known. To make (6.5) a useful boundary condition for θ^+, a relationship between ϕ^+ and ϕ_1 must be found. By equating differential arc-length along each side of OC we find

$$d\bar{s} = \frac{d\bar{\phi}_1}{\bar{q}_1} = \frac{d\bar{\phi}_2}{\bar{q}_2} \quad \text{for} \quad \psi = 0. \tag{6.9}$$

Using (5.1a) and (4.14) with $\tilde{\psi} = 0$,

$$\bar{q}_1 = q_{\infty 1}e^{\mu_1 Q^* + \cdots} \approx q_{\infty 1} + o(1), \tag{6.10a}$$

and

$$\bar{q}_2 = q_{\infty 2}e^{a_0\tilde{\phi}^{\alpha/\pi}} + \dots. \tag{6.10b}$$

Substituting in (6.9), we obtain to first order after integrating and choosing $\phi_1 = 0$ when $\phi_2 = 0$,

$$\phi_1 = \frac{e^{-a_0}}{1-(\alpha/\pi)}\frac{\mu_2^{1-(\alpha/\pi)}}{\mu_1}\phi_2^{1-(\alpha/\pi)} \quad \text{for} \quad \phi_1,\ \phi_2 > 0. \tag{6.11}$$

Comparing this with (6.2) when $\psi = 0$, $\phi_2 > 0$, shows that to first order

$$\phi^+ = \phi_1 \quad \text{on} \quad \psi = 0 \quad \text{for} \quad \phi^+, \phi_1 > 0. \tag{6.12}$$

Therefore, the boundary condition (6.5) may be written using (6.8)

$$\theta^+(\phi^+,0) = f(\phi^+). \tag{6.13}$$

Finally as $|w^+| \to \infty$, (6.1) must merge with the expansion of $\tilde{\Gamma}$ when $|\tilde{w}| \to 0$. Since the first two terms of (6.1) are the same as those in (4.14), we require $\mu_1\theta^+$ to match with minus the imaginary part of the third term in (4.14),† i.e.

$$\mu_1\theta^+ \sim -\mathrm{Im}\{a_1(e^{\pi i}\tilde{w})^{1-(\alpha/\pi)}\} \quad \text{for} \quad |w^+| \to \infty. \tag{6.14}$$

† It has been assumed that higher order terms of (4.4b) will contribute terms of smaller order to θ^+.

Using (6.2) to rewrite (6.14) in terms of w^+ yields

$$\begin{aligned}\theta^+ &\sim -\operatorname{Im}\{a_1 e^{a_0(1-[\alpha/\pi])} e^{i(\pi-\alpha)} w^+\} \\ &\sim -a_1 e^{a_0(1-[\alpha/\pi])}[\phi^+ \sin(\pi-\alpha) + \psi^+ \cos(\pi-\alpha)] \\ &\sim \tfrac{1}{2}(\phi^+ - \psi^+ \cot\alpha) \quad \text{for} \quad |w^+| \to \infty, \qquad (6.15)\end{aligned}$$

where we have used (4.15). The absence of μ_1 indicates that the expansion variables were chosen correctly.

The boundary conditions (6.4), (6.13) and (6.15) are sufficient to determine a unique harmonic function which does not grow more rapidly than (6.15) as $|w^+| \to \infty$. To see this more clearly, note that for $\psi^+ = 0$, $\phi^+ \to \infty$, we may use (6.13) and (5.20) to find

$$\theta^+(\phi^+, 0) = f(\phi^+) \sim \tfrac{1}{2}\phi^+ + \tfrac{1}{2}([1/\pi]\ln 2 - C) \quad \text{for} \quad \phi^+ \to \infty. \qquad (6.16)$$

An asymptotic solution for θ^+ when $|w^+| \to \infty$ is given by

$$\theta^+(\phi^+, \psi^+) \sim \tfrac{1}{2}(\phi^+ - \psi^+ \cot\alpha) + \frac{1}{2(\pi-\alpha)}([1/\pi]\ln 2 - C)(\arg w^+ + \pi - \alpha)$$
$$\text{for} \quad |w^+| \to \infty. \qquad (6.17)$$

It is easy to verify by inspection that (6.4), (6.15) and (6.16) are all satisfied by (6.17). Furthermore, since the right-hand side of (6.17) is harmonic, we may seek a new harmonic function for the difference between θ^+ and the right-hand side of (6.17). This new function must satisfy (6.4), vanish at infinity and satisfy an inhomogeneous boundary condition along $\psi^+ = 0$, $\phi^+ > 0$. Such a Dirichlet problem may always be solved. Note that θ^+ depends on $\tilde{\Gamma}$ only through the single constant a_0 in (6.2).

In the special case $\alpha = \frac{1}{2}\pi$, $f(\phi^+) \equiv \frac{1}{2}\phi^+$ (see p. 595) and

$$\theta^+(\phi^+, \psi^+) \equiv \tfrac{1}{2}\phi^+, \qquad (6.18)$$

with

$$\Gamma^+(w^+) \equiv -\tfrac{1}{2} i w^+. \qquad (6.19)$$

When (6.19) is inserted in (6.1), and the resulting equation expressed in terms of $\tilde{w}$ using (6.2), we obtain the first three terms in (4.14). Therefore, for $\alpha = \frac{1}{2}\pi$, $\tilde{\Gamma}$ may be used to compute the first-order streamline curvature correctly to $O(\mu_1)$ throughout the external flow.

7. Summary and conclusions

Although no numerical results have been obtained in this paper (the reader is referred to the A–P paper for the numerical solution when $\alpha = \frac{1}{2}\pi$), a number of interesting conclusions have been reached. On the assumption that $\tilde{\Gamma}$ is a uniformly valid first-order solution correct to $O(1)$ in the external flow, we have been able to formulate in a consistent way the boundary-value problems for the inner solutions in the jet and the external flow; in the case of the jet, an explicit solution was found using conformal mapping. The matching of various solutions in overlapping regions and along common boundaries leaves little doubt that

$\tilde{\Gamma}$ is uniformly valid as assumed. A summary of the important results is given here.

The jet thickness remains constant ($=d_0$) to first order throughout its course.

For $0 < \alpha < \pi$, the ratio of the jet thickness to the radius of curvature of the jet centreline $[\psi = O(1)]$ is of $O(\mu_1)$ uniformly to first order. For $\alpha > \frac{1}{2}\pi$, the curvature (non-dimensionalized with respect to d_0) of the vortex sheet OC is infinite (but integrable) at distances of $o[d_0 \mu_1{}^{2\alpha/(2\alpha-\pi)}]$ from the jet exit when $\mu_1 \to 0$.

For $\alpha \neq \frac{1}{2}\pi$, $\hat{Q}$ (of the thin jet approximation) fails to give the correct speed distribution inside the jet ($0 < \psi < 1$) at distances of $O(d_0)$ from the jet exit. An important point is that (3.7) does give the correct limit when $\hat{\phi} \to 0$ along $\psi \equiv 0$, i.e. $\hat{Q} \to -\frac{1}{2}$ in agreement with the boundary condition (5.9). Thus, $\hat{Q}$ is valid all along OC (and also along the free streamline AB) to first order, but not uniformly in the interior. This accounts for the validity of $\tilde{\Gamma}$ in the region adjacent to that where the thin jet approximation fails. The streamline deflexion $\hat{\theta}$ is uniformly valid for all α, but cannot be used to determine the streamline curvature in the jet correctly to $O(\mu_1)$ at distances of $O(d_0)$ from the jet exit An inner solution must be used for the curvature in these cases.

For $\alpha \equiv \frac{1}{2}\pi$, the thin jet approximation is uniformly valid to first order throughout the jet and can be used to compute the streamline curvature to $O(\mu_1)$ uniformly.

For $0 < \alpha < \pi$, $\tilde{\Gamma}$ is a uniformly valid first-order solution correct to $O(1)$ in the external flow. When $\alpha \neq \frac{1}{2}\pi$, the first-order streamline curvature given by $\tilde{\Gamma}$ is incorrect at distances of $O(d_0)$ from the jet exit, and an inner solution is necessary to correct it to $O(\mu_1)$. For $\alpha \equiv \frac{1}{2}\pi$, $\tilde{\Gamma}$ gives the streamline curvature correctly to $O(\mu_1)$ uniformly.

The pressure at distances of $O(d_0)$ from the jet exit in the external flow is constant to first order and equal to the stagnation pressure of the external flow.

Most results given here would be drastically altered if real fluids with small viscosities had been considered, and this must not be overlooked. Near the jet exit, however, where the cumulative effects of viscosity will be small, some experimental verification of these results may be possible.

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Appendix

1. The thin jet approximation

From (3.2*a*) and (3.7), the first-order speed distribution inside the jet is

$$\bar{q}_1/q_{\infty 1} = e^{Q_1} = \exp\{\mu_1 \hat{Q} + \ldots\} \approx 1 + \mu_1 \hat{Q} + \ldots \approx 1 + \mu_1 \hat{\theta}'(\hat{\phi})[\psi - 1]. \quad \text{(A 1.1)}$$

Noting that the first-order curvature of any streamline in the jet is

$$1/\bar{R}_1 = \bar{q}_1(\partial\theta_1/\partial\bar{\phi}_1) \approx (\mu_1 q_{\infty 1}/m)\hat{\theta}'(\hat{\phi}) \approx \mu_1 \hat{\theta}'(\hat{\phi})/d_0, \quad \text{(A 1.2)}$$

we obtain on eliminating $\mu_1\hat{\theta}'$ between (A 1.1) and (A 1.2)

$$\bar{q}_1/q_{\infty 1} = 1+(d_0/\bar{R}_1)[\psi-1]. \quad \text{(A 1.3)}$$

Along a streamline in the jet, $\bar{q}_1 = \partial\bar{\psi}/\partial\bar{n}$, where $\bar{n}$ is the distance measured normal to the streamlines in the direction of $\bar{\psi}$ increasing. Therefore, to first order

$$\partial\bar{\psi}/\partial\bar{n} \approx q_{\infty 1}. \quad \text{(A 1.4)}$$

On integrating and choosing $\psi = 0$ when $n = 0$, we obtain

$$\psi = \bar{n}q_{\infty 1}/m \approx \bar{n}/d_0 \quad \text{for} \quad 0 \leqslant \bar{n} \leqslant d_0. \quad \text{(A 1.5)}$$

Substituting for ψ in (A 1.3),

$$\bar{q}_1/q_{\infty 1} = 1+\frac{\bar{n}-d_0}{\bar{R}_1} \approx \frac{1-d_0/\bar{R}_1}{1-\bar{n}/\bar{R}_1} \quad \text{when} \quad d_0/\bar{R}_1 \ll 1. \quad \text{(A 1.6)}$$

This is the usual form of the thin jet approximation.

2. Proof that $\hat{\theta}''(0) = 0$ for $\alpha = \frac{1}{2}\pi$

On crossing OC in the outer region, the deflexion must be continuous. Using (2.16a), (3.2b), (3.4) and (4.4b), we obtain to first order

$$\hat{\theta}[\hat{\phi}(\sigma)] = \tilde{\theta}[\tilde{\phi}(\sigma), 0]. \quad \text{(A 2.1)}$$

Since $\hat{Q}$ and $\tilde{Q}$ are valid all along OC, we may relate $\hat{\phi}$ and $\tilde{\phi}$ by equating arc-length on each side [see (6.9)]. The result will be the same as (6.11) with $\alpha = \frac{1}{2}\pi$, i.e.

$$\hat{\phi} = 2\,e^{-a_0}\tilde{\phi}^{\frac{1}{2}}+\ldots \quad \text{for} \quad \hat{\phi}, \tilde{\phi} > 0. \quad \text{(A 2.2)}$$

Differentiate both sides of (A 2.1) with respect to $\hat{\phi}$ using (4.10) and (A 2.2) to obtain

$$\hat{\theta}'(\hat{\phi}) = \frac{d\tilde{\phi}}{d\hat{\phi}}\frac{\partial\tilde{\theta}}{\partial\tilde{\phi}} \approx -e^{a_0}\tilde{\phi}^{\frac{1}{2}}\sinh\tilde{Q}(\tilde{\phi},0),$$
$$\approx -\tfrac{1}{2}e^{a_0}[e^{a_0}\tilde{\phi}-e^{-a_0}], \quad \text{(A 2.3)}$$

where the last step depends on the result obtained from (4.14),

$$\tilde{Q}(\tilde{\phi},0) = \tfrac{1}{2}\ln\tilde{\phi}+a_0+O(\tilde{\phi}) \quad \text{for} \quad \tilde{\phi}\to 0. \quad \text{(A 2.4)}$$

Differentiate (A 2.3) with respect to $\hat{\phi}$

$$\hat{\theta}''(\hat{\phi}) \approx -\tfrac{1}{2}e^{a_0}\frac{d\tilde{\phi}}{d\hat{\phi}}e^{a_0} \approx -\tfrac{1}{2}e^{3a_0}\tilde{\phi}^{\frac{1}{2}} \approx -\tfrac{1}{4}e^{4a_0}\hat{\phi}, \quad \text{(A 2.5)}$$

which establishes the result.

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